

Coherent dynamics of radiating atomic systems in pseudospin representation

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Abstract

The aim of this review is twofold. First, a general approach is presented allowing for a unified description of dynamics in radiating systems of different nature. Both atomic systems as well as spin assemblies can be treated in the frame of the same mathematical method based on pseudospin (or spin) representation of evolution equations. The approach is applicable to all stages of radiation dynamics, including the most difficult initial quantum stage, where coherence is not yet developed. This makes it possible to study the process of coherent self-organization from the chaotic quantum stage. Second, the approach is illustrated by applying it for the description of several coherent phenomena. Different types of superradiance are characterized: pure superradiance, triggered superradiance, pulsing and punctuated superradiance. The theory is presented of such interesting effects as triggering dipolar waves, turbulent photon filamentation, collective liberation of light, pseudospin atomic squeezing, and operator entanglement production.

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1 Introduction

Coherent radiation from atomic and molecular systems is well studied, being the basis of laser radiation. The theory of this radiation, occurring in optical or infrared diapason, is thoroughly expounded in many books (see, e.g. [1–5]). Another type of systems whose collective radiation has recently attracted much attention are spin assemblies [6]. Atomic and spin systems, being rather different in nature, are usually described by different types of equations. Dynamical physical processes in these systems are really quite different. However, it is possible to develop a mathematical approach that would allow for a similar description of both these system types. Such an approach is presented in this review, where we concentrate on the dynamics of atomic systems, illustrating the approach by the description of several coherent phenomena.

A problem of great interest is the self-organization of coherence from initially chaotic quantum fluctuations. The developed approach gives the possibility of analyzing such a self-organization in detail. Briefly speaking, the basic mathematical points of the approach are: pseudospin representation of evolution equations, stochastic quantization, scale separation, transverse mode expansion, and probabilistic pattern selection.

As illustrations of the approach, several nontrivial phenomena will be treated, such as triggering dipolar waves, different kinds of superradiance (pure superradiance, triggered superradiance, pulsing and punctuated superradiance), turbulent photon filamentation, collective liberation of light, pseudospin atomic squeezing, and operator entanglement production.

The main point, emphasized throughout the paper, is the generality of the used mathematical techniques that can be applied to any kind of evolution equations describing radiating systems. It is evident that the equations of motion for spin systems are written for the spin degrees of freedom. The pseudospin representation for atomic systems makes it straightforward to employ the same mathematical techniques for spin as well as for atomic radiating systems.

This review is planned for the Special Issue devoted to the memory of Igor V. Yevseyev. Igor was my friend for many years and I had a pleasure of discussing with him various scientific and non-scientific problems. My memory about Igor is interconnected with those problems we discussed. Exactly this interconnection dictated the choice of the material for this review, in which I included the material I had discussed with Igor.

2 Equations in pseudospin representation

In order to reduce the equations for radiating atomic systems to the form that would also be convenient for treating spin systems, it is natural to resort to pseudospin representation and eliminate the field degrees of freedom [7,8]. In addition, aiming at accurately characterizing the self-organization of coherence from chaos, it is necessary to start with a microscopic picture.

Let us consider N atoms (or molecules) resonantly interacting with electromagnetic field. The general form of the Hamiltonian is

$$\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_{af} + \hat{H}_{mf} . \quad (1)$$

The first term

$$\hat{H}_a = \sum_{j=1}^N \omega_0 \left(\frac{1}{2} + S_j^z \right) \quad (2)$$

corresponds to resonant atoms with the transition frequency ω_0 ; the pseudospin operator S_j^z characterizes interlevel electronic transitions of a j th atom. Considering here two-level atoms,

we deal with the pseudospin operators of spin one-half. Atoms with a larger number of resonant levels would require the use of higher-order pseudospin operators. The operators are called pseudospin because they satisfy the spin algebra, but do not describe real spins, characterizing instead interlevel transitions.

Here and in what follows, the system of units is employed where the Planck constant is set to unity.

The second term in equation (1) defines the energy of radiated electromagnetic field

$$\hat{H}_f = \frac{1}{8\pi} \int (\mathbf{E}^2 + \mathbf{H}^2) d\mathbf{r} , \quad (3)$$

where \mathbf{E} is electric field and \mathbf{H} is magnetic field represented through vector potential \mathbf{A} ,

$$\mathbf{H} = \nabla \times \mathbf{A} . \quad (4)$$

The vector potential is assumed to satisfy the Coulomb gauge calibration

$$\nabla \cdot \mathbf{A} = 0 . \quad (5)$$

The third term describes the atom-field interaction

$$\hat{H}_{af} = - \sum_{j=1}^N \left(\frac{1}{c} \mathbf{J}_j \cdot \mathbf{A}_j + \mathbf{P}_j \cdot \mathbf{E}_{0j} \right) , \quad (6)$$

where dipolar transitions are assumed, \mathbf{E}_{0j} is an external electric field, the current operator is

$$\mathbf{J}_j = -i\omega_0 (\mathbf{d}^* S_j^- - \mathbf{d} S_j^+) \quad (7)$$

and the polarization operator is

$$\mathbf{P}_j = \mathbf{d}^* S_j^- + \mathbf{d} S_j^+ , \quad (8)$$

with the ladder pseudospin operators

$$\mathbf{S}_j \equiv S_j^x \pm i S_j^y ,$$

and \mathbf{d} being a transition dipole. The notation

$$\mathbf{A}_j \equiv \mathbf{A}(\mathbf{r}_j, t) , \quad \mathbf{E}_{0j} \equiv \mathbf{E}_0(\mathbf{r}_j, t)$$

is used.

The last term in equation (1) describes the interaction of the radiated field with the matter surrounding atoms. This term is absent when atoms are in vacuum. But if atoms are immersed into some kind of matter, the interaction term is

$$\hat{H}_{mf} = - \frac{1}{c} \int \mathbf{j}_{mat} \cdot \mathbf{A} d\mathbf{r} , \quad (9)$$

where \mathbf{j}_{mat} is the density current in the matter.

The evolution equations are prescribed by the Heisenberg equations of motion, with the corresponding commutation relations. The electromagnetic operators satisfy the relations

$$[E^\alpha(\mathbf{r}, t), A^\beta(\mathbf{r}', t)] = 4\pi i c \delta_{\alpha\beta}(\mathbf{r} - \mathbf{r}') , \quad [A^\alpha(\mathbf{r}, t), H^\beta(\mathbf{r}', t)] = 0 ,$$

$$[E^\alpha(\mathbf{r}, t), H^\beta(\mathbf{r}', t)] = -4\pi i c \sum_{\gamma} \varepsilon_{\alpha\beta\gamma} \frac{\partial}{\partial r^\gamma} \delta(\mathbf{r} - \mathbf{r}') ,$$

in which ε_{ijk} is the unitary antisymmetric tensor, c is light velocity, and the transverse delta function is defined as

$$\delta_{\alpha\beta}(\mathbf{r}) = \frac{2}{3} \delta_{\alpha\beta} \delta(\mathbf{r}) - \frac{1}{4\pi} D_{\alpha\beta}(\mathbf{r}) ,$$

with the dipolar tensor

$$D_{\alpha\beta}(\mathbf{r}) \equiv \frac{1}{r^3} (\delta_{\alpha\beta} - 3n^\alpha n^\beta) ,$$

where

$$r \equiv |\mathbf{r}| , \quad \mathbf{n} \equiv \frac{\mathbf{r}}{r} .$$

The pseudospin operators obey the spin algebra

$$[S_j^-, S_i^+] = -2\delta_{ij} S_j^z , \quad [S_j^-, S_i^z] = \delta_{ij} S_j^- , \quad [S_j^+, S_i^z] = -\delta_{ij} S_j^+ .$$

The Heisenberg equations for the field variables yield the Maxwell equations

$$\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \frac{4\pi}{c} \mathbf{j} , \quad \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} , \quad (10)$$

from where, with the Coulomb calibration (5), the wave equation follows:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\frac{4\pi}{c} \mathbf{j} , \quad (11)$$

with the density of current

$$j^\alpha(\mathbf{r}, t) = \sum_{\beta} \left[\sum_{i=1}^N \delta_{\alpha\beta}(\mathbf{r} - \mathbf{r}_i) J_i^\beta(t) + \int \delta_{\alpha\beta}(\mathbf{r} - \mathbf{r}') j_{mat}^\beta(\mathbf{r}', t) d\mathbf{r}' \right] . \quad (12)$$

The solution to equation (11) reads

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_{vac}(\mathbf{r}, t) + \frac{1}{c} \int \mathbf{j} \left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right) \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} , \quad (13)$$

where \mathbf{A}_{vac} is the vacuum vector potential that is a solution to the equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A}_{vac} = 0 . \quad (14)$$

The Heisenberg equations for the pseudospin variables give

$$\begin{aligned} \frac{dS_j^-}{dt} &= -i\omega_0 S_j^- + 2S_j^z (k_0 \mathbf{d} \cdot \mathbf{A}_j - i\mathbf{d} \cdot \mathbf{E}_{0j}) , \\ \frac{dS_j^z}{dt} &= -S_j^+ (k_0 \mathbf{d} \cdot \mathbf{A}_j - i\mathbf{d} \cdot \mathbf{E}_{0j}) - S_j^- (k_0 \mathbf{d}^* \cdot \mathbf{A}_j + i\mathbf{d}^* \cdot \mathbf{E}_{0j}) , \end{aligned} \quad (15)$$

where $k_0 = \omega_0/c$.

The vector potential (13), with the current density (12), can be written, excluding self-action, as

$$\mathbf{A} = \mathbf{A}_{vac} + \mathbf{A}_{rad} + \mathbf{A}_{dip} + \mathbf{A}_{mat} . \quad (16)$$

The first term here is due to radiating atoms,

$$\mathbf{A}_{rad}(\mathbf{r}_i, t) = \sum_{j(\neq i)}^N \frac{2}{3cr_{ij}} \mathbf{J}_j \left(t - \frac{r_{ij}}{c} \right) , \quad (17)$$

where

$$r_{ij} \equiv |\mathbf{r}_{ij}| , \quad \mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j .$$

Using the form of current (7), we find

$$\mathbf{A}_{rad}(\mathbf{r}_i, t) = -i \sum_{j(\neq i)}^N \frac{2k_0}{3r_{ij}} \left[\mathbf{d}^* S_j^- \left(t - \frac{r_{ij}}{c} \right) - \mathbf{d} S_j^+ \left(t - \frac{r_{ij}}{c} \right) \right] . \quad (18)$$

The second term in equation (16) is caused by induced atomic dipoles,

$$A_{dip}^\alpha(\mathbf{r}, t) = - \sum_{j=1}^N \sum_{\beta} \int \frac{D_{\alpha\beta}(\mathbf{r}' - \mathbf{r}_j)}{4\pi c |\mathbf{r} - \mathbf{r}'|} J_j^\beta \left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right) d\mathbf{r}' . \quad (19)$$

And the last term

$$A_{mat}^\alpha(\mathbf{r}, t) = \sum_{\beta} \int \frac{\delta_{\alpha\beta}(\mathbf{r}' - \mathbf{r}'')}{c |\mathbf{r} - \mathbf{r}'|} j_{mat}^\beta \left(r'', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right) d\mathbf{r}' d\mathbf{r}'' \quad (20)$$

corresponds to the vector potential created by the current in the matter.

Notice that, due to the dependence on the variable $t - r/c$, we have

$$\left(\frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right) S_j^\pm \left(t - \frac{r}{c} \right) = 0 . \quad (21)$$

We assume that electromagnetic fields do not strongly disturb atomic level structure, so that

$$\frac{|\mathbf{d} \cdot \mathbf{E}|}{\omega_0} \ll 1 , \quad \frac{|\mathbf{d} \cdot \mathbf{E}_0|}{\omega_0} \ll 1 . \quad (22)$$

Then the retardation effects can be described in the Born approximation. Under condition (22), from equations (15) it follows

$$\frac{\partial}{\partial r} S_j^\pm \left(t - \frac{r}{c} \right) = \mp i k_0 S_j^\pm \left(t - \frac{r}{c} \right) . \quad (23)$$

Setting the retardation condition

$$S_j^\alpha(t) = 0 \quad (t < 0) , \quad (24)$$

we finally get in the Born approximation

$$S_j^- \left(t - \frac{r}{c} \right) = S_j^-(t) \Theta(ct - r) e^{ik_0 r} , \quad S_j^z \left(t - \frac{r}{c} \right) = S_j^z(t) \Theta(ct - r) . \quad (25)$$

3 Self-action of a radiating atom

Strictly speaking, substituting the current density (12) into the integral in equation (13), one meets the terms corresponding to the atomic self-action, which can be treated as follows. The vector potential generated by a single atom is

$$A_s^\alpha(\mathbf{r}, t) = \frac{1}{c} \sum_\beta \int \frac{\delta_{\alpha\beta}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} J^\beta \left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right) d\mathbf{r}' , \quad (26)$$

with the current

$$\mathbf{J} \left(t - \frac{r}{c} \right) = i\omega_0 [\mathbf{d}S^+(t) e^{-ik_0r} - \mathbf{d}^* S^-(t) e^{ik_0r}] \Theta(ct - r) ,$$

where $S^\alpha(t) \equiv S^\alpha(0, t)$. At small distance, such that $k_0r \ll 1$, one may write $e^{ik_0r} \simeq 1 + ik_0r$. Substituting the transverse δ -function into the vector potential (26), we keep in mind that averaging the dipolar tensor over spherical angles gives

$$\int D_{\alpha\beta}(\mathbf{r}) d\Omega(\mathbf{r}) = 0 .$$

Then, for $k_0r \ll 1$, the vector potential (26) becomes

$$\mathbf{A}_s(\mathbf{r}, t) \simeq \frac{2}{3} k_0^2 [\mathbf{d}S^+(t) + \mathbf{d}^*S^-(t)] + i \frac{2k_0}{3r} [\mathbf{d}S^+(t) - \mathbf{d}^*S^-(t)] .$$

To avoid the divergence in the term $1/r$, one can average it between the electron wavelength $\lambda_e = 2\pi\hbar/mc$, with m being the electron mass, and the radiation wavelength $\lambda_0 = 2\pi/k_0$. Taking into account that $\lambda_e \ll \lambda_0$, we have

$$\frac{1}{\lambda_0 - \lambda_e} \int_{\lambda_e}^{\lambda_0} \frac{dr}{r} = \frac{k_0}{2\pi} \ln \left(\frac{mc^2}{\hbar\omega_0} \right) .$$

Then for the self-acting vector potential, we get

$$\mathbf{A}_s(0, t) = \frac{2}{3} k_0^2 [\mathbf{d}S^+(t) + \mathbf{d}^*S^-(t)] + \frac{ik_0}{3\pi} \ln \left(\frac{mc^2}{\hbar\omega_0} \right) [\mathbf{d}S^+(t) - \mathbf{d}^*S^-(t)] . \quad (27)$$

Substituting this into the evolution equations for the case of a single atom, we employ the properties of spin one-half operators. Then we come to the equations for a single atom

$$\frac{dS^-}{dt} = -i(\omega_0 - \delta_L - i\gamma_0)S^- + \frac{\mathbf{d}^2}{|\mathbf{d}|^2} (\gamma_0 + i\delta_L)S^+ , \quad \frac{dS^z}{dt} = -2\gamma_0 \left(\frac{1}{2} + S^z \right) , \quad (28)$$

in which the notations for the natural width

$$\gamma_0 \equiv \frac{2}{3} |\mathbf{d}|^2 k_0^3 \quad (29)$$

and the Lamb shift

$$\delta_L \equiv \frac{\gamma_0}{2\pi} \ln \left(\frac{mc^2}{\hbar\omega_0} \right) \quad (30)$$

are introduced. The solutions to equations (28), keeping in mind that $\gamma_0 \ll \omega_0$ and $\delta_L \ll \omega_0$, are

$$S^-(t) = S^-(0) \exp \{ -i(\omega_0 - \delta_L)t - \gamma_0 t \} ,$$

$$S^z(t) = -\frac{1}{2} + \left[\frac{1}{2} + S^z(0) \right] \exp(-2\gamma_0 t) .$$

Thus, the self-action of a radiating atom leads to the appearance of the attenuation in the dynamics of the pseudospin operators and to the Lamb frequency shift. The latter can always be included in the definition of the transition frequency ω_0 . Taking into consideration the attenuation, one usually generalizes the equations of motion by including γ_2 , instead of γ_0 , for S_i^- and inserting γ_1 , instead of $2\gamma_0$, for S_i^z .

4 Stochastic mean-field quantization

Expressing the vector potential (16) through the pseudospin variables, we obtain the pseudospin equations (15) involving only the pseudospin degrees of freedom. However, these equations are not yet closed, containing the products of the pseudospin operators. If we employ the semiclassical approximation, this would eliminate quantum fluctuations, which would make it impossible to describe the initial stage of radiation, when coherence is not yet developed. In order to simplify the equations by means of a kind of a mean-field approximation, at the same time not losing the influence of quantum fluctuations, we use the stochastic mean-field quantization [8].

We notice that in equations (15) there are the terms with different properties. The combination

$$\xi = 2k_0 \mathbf{d} \cdot (\mathbf{A}_{vac} + \mathbf{A}_{dip} + \mathbf{A}_{mat}) \quad (31)$$

describes short-range fast fluctuations. While the remaining terms containing the pseudospin variables in the radiation vector potential (18) are of long-range nature. Thus, it is admissible to distinguish two types of the variables. One of them, ξ can be treated as a stochastic variable, while the remaining set

$$\tilde{S} \equiv \{ \mathbf{S}_j : j = 1, 2, \dots, N \} \quad (32)$$

is treated as a collection of usual spin operators. Thus all quantities in equations (15) are functions of two variables, which can be denoted as $f(\tilde{S}, \xi)$.

Having two types of the variables, it is natural to introduce two different averaging procedures. One is the spin averaging

$$\langle f(\tilde{S}, \xi) \rangle \equiv \text{Tr} \hat{\rho} f(\tilde{S}, \xi) , \quad (33)$$

with a statistical operator $\hat{\rho}$ and the trace over the spin degrees of freedom. And the other is the stochastic averaging

$$\langle \langle f(\tilde{S}, \xi) \rangle \rangle \equiv \int f(\tilde{S}, \xi) \mathcal{D}\xi , \quad (34)$$

with the functional integration over the stochastic variable ξ .

Since the vector potential (18) characterizes long-range interactions, decaying as $1/r$, it is possible to use the mean-field decoupling with respect to the spin averaging:

$$\langle S_i^\alpha S_j^\beta \rangle = \langle S_i^\alpha \rangle \langle S_j^\beta \rangle \quad (i \neq j) , \quad (35)$$

where the stochastic variable ξ is kept untouched.

Accomplishing the spin averaging of equations (15), we define the *transition function*

$$u(\mathbf{r}_j, t) \equiv 2\langle S_j^-(t) \rangle , \quad (36)$$

coherence intensity

$$w(\mathbf{r}_j, t) \equiv \frac{2}{N} \sum_{i \neq j}^N \langle S_i^+(t) S_j^-(t) + S_j^+(t) S_i^-(t) \rangle , \quad (37)$$

and the population imbalance

$$s(\mathbf{r}_j, t) \equiv 2\langle S_j^z(t) \rangle . \quad (38)$$

To simplify the presentation of the resulting equations, we introduce the effective external force, due to the external field,

$$f_0(\mathbf{r}, t) \equiv -2i\mathbf{d} \cdot \mathbf{E}_0(\mathbf{r}, t) \quad (39)$$

and the effective radiation force

$$f_{rad}(\mathbf{r}, t) \equiv 2k_0 \langle \mathbf{d} \cdot \mathbf{A}(\mathbf{r}, t) \rangle , \quad (40)$$

caused by atomic interactions through the common radiation field. Passing from the summation over atoms to the spatial integration by means of the replacement

$$\sum_{j=1}^N \longrightarrow \rho \int d\mathbf{r} \quad \left(\rho \equiv \frac{N}{V} \right) ,$$

with the integration over the whole atomic system, we get the effective radiation force

$$f_{rad}(\mathbf{r}, t) = -i\gamma_0 \rho \int [G(\mathbf{r} - \mathbf{r}', t) u(\mathbf{r}', t) - \mathbf{e}_d^2 G^*(\mathbf{r} - \mathbf{r}', t) u^*(\mathbf{r}', t)] d\mathbf{r}' , \quad (41)$$

where $\mathbf{e}_d \equiv \mathbf{d}/|\mathbf{d}|$ and the transfer function is

$$G(\mathbf{r}, t) \equiv \Theta(ct - r) \frac{\exp(ik_0 r)}{k_0 r} .$$

The total effective force, acting on the spin variables, is the sum

$$f(\mathbf{r}, t) \equiv f_0(\mathbf{r}, t) + f_{rad}(\mathbf{r}, t) + \xi(\mathbf{r}, t) . \quad (42)$$

Finally, we come to the equations for the transition function,

$$\frac{\partial u}{\partial t} = -(i\omega_0 + \gamma_2) u + f s , \quad (43)$$

for the coherence intensity,

$$\frac{\partial w}{\partial t} = -2\gamma_2 w + (u^* f + f^* u) s , \quad (44)$$

and for the population imbalance,

$$\frac{\partial s}{\partial t} = -\frac{1}{2} (u^* f + f^* u) - \gamma_1 (s - \zeta) , \quad (45)$$

where ζ is an equilibrium population imbalance for a single atom. As usual, these equations are assumed to be complimented by the corresponding initial conditions and, if necessary, boundary conditions.

5 Scale separation approach

The following analysis of equations (45) can be done by using the scale separation approach [7–10] that is a generalization to stochastic differential equations with multiple scales of the Krylov-Bogolubov averaging method [11]. Partial differential equations can also be treated by this method [8, 12]. Here we delineate the idea of the scale separation approach, keeping in mind equations (45).

The separation of scales is based on the existence of small parameters. Thus, the influence of the external field is small according to equation (22). The attenuations are also assumed to be small, so that

$$\frac{\gamma_0}{\omega_0} \ll 1, \quad \frac{\gamma_1}{\omega_0} \ll 1, \quad \frac{\gamma_2}{\omega_0} \ll 1. \quad (46)$$

Equations (45) can be written in the form

$$\frac{\partial u}{\partial t} = f_u, \quad \frac{\partial w}{\partial t} = f_w, \quad \frac{\partial s}{\partial t} = f_s, \quad (47)$$

in which

$$f_\alpha = f_\alpha(u, w, s, \xi, t) \quad (\alpha = u, w, s),$$

The right-hand sides here are such that, if all small parameters tend to zero, then

$$f_w \rightarrow 0, \quad f_s \rightarrow 0, \quad (48)$$

while f_u remains finite. This means that the functions w and s are quasi-integrals of motion, or slow variables, while u is a fast variable. The equation for u is solved, with the slow variables w and s kept fixed, which defines $u = u(w, s, \xi, t)$. Then this solution is substituted into the equations for the slow variables, with averaging their right-hand sides over the fast variable and over the stochastic variable, according to the rule

$$\overline{f}_\alpha(w, s) \equiv \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \langle \langle f_\alpha(u, w, s, \xi, t) \rangle \rangle dt. \quad (49)$$

This results in the guiding-center equations for the slow variables:

$$\frac{\partial w}{\partial t} = \overline{f}_w, \quad \frac{\partial s}{\partial t} = \overline{f}_s. \quad (50)$$

If necessary, it is possible to find corrections to the guiding centers [8–10].

6 Magnetic dipole transitions

In the previous sections, it has been assumed that the resonant atoms experience electric dipole transitions. Now we consider magnetic dipole transitions. We show that, despite their difference, both these transitions lead to the same kind of evolution equations. This consideration is also of importance for answering whether spin systems could demonstrate the occurrence of superradiance in the same way as atoms. It turns out that there is a principal difference between atomic and spin systems, since the latter, in addition to the interactions through the common radiation field, possess rather strong direct dipole spin interactions destroying coherence [13–16].

In the case of magnetic dipole transitions, the atom-field interaction is given by the Hamiltonian

$$\hat{H}_{af} = - \sum_{j=1}^N \mathbf{M}_j \cdot \mathbf{B}_j , \quad (51)$$

where $\mathbf{B}_j = \mathbf{B}(\mathbf{r}_j, t)$, the magnetic moment can be written [8, 14] as

$$\mathbf{M}_j = \vec{\mu}^* S_j^- + \vec{\mu} S_j^+ + \vec{\mu}_0 S_j^z , \quad (52)$$

and the total magnetic field is the sum

$$\mathbf{B} = \mathbf{H}_0 + \mathbf{H} + \mathbf{H}_{vac} + \mathbf{H}_{mat} \quad (53)$$

of an external field, radiation field, vacuum field, and the field of the matter which the atoms are inserted in.

The Heisenberg equations of motion yield

$$\begin{aligned} \frac{dS_j^-}{dt} &= -i\omega_0 S_j^- + i(\vec{\mu}_0 S_j^- - 2\vec{\mu} S_j^z) \cdot \mathbf{B}_j , \\ \frac{dS_j^z}{dt} &= i(\vec{\mu} S_j^+ - \vec{\mu}^* S_j^-) \cdot \mathbf{B}_j . \end{aligned} \quad (54)$$

The total vector potential is the sum

$$\mathbf{A} = \mathbf{A}_{rad} + \mathbf{A}_{vac} + \mathbf{A}_{mat} . \quad (55)$$

The first term, caused by atomic radiation, is

$$\mathbf{A}_{rad}(\mathbf{r}, t) = \frac{1}{c} \int \mathbf{j} \left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right) \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} , \quad (56)$$

with the current density

$$\mathbf{j}(\mathbf{r}, t) = -c \sum_{j=1}^N \mathbf{M}_j(t) \times \nabla \delta(\mathbf{r} - \mathbf{r}_j) . \quad (57)$$

Taking into account magnetic momentum (52) makes it possible to rewrite the vector potential (56) in the form

$$\mathbf{A}_{rad} = \mathbf{A}_+ + \mathbf{A}_- + \mathbf{A}_z , \quad (58)$$

in which

$$\mathbf{A}_+(\mathbf{r}_i, t) = - \sum_{j(\neq i)}^N \frac{1}{r_{ij}^2} \left(1 + \frac{r_{ij}}{c} \frac{\partial}{\partial t} \right) \mathbf{n}_{ij} \times \vec{\mu} S_j^+ \left(t - \frac{r_{ij}}{c} \right) , \quad (59)$$

\mathbf{A}_- is the Hermitian conjugate to \mathbf{A}_+ ,

$$\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j , \quad \mathbf{n}_{ij} \equiv \frac{\mathbf{r}_{ij}}{r_{ij}} \quad r_{ij} \equiv |\mathbf{r}_{ij}| ,$$

and the last term of equation (58) is

$$\mathbf{A}_z(\mathbf{r}_i, t) = - \sum_{j(\neq i)}^N \frac{\mathbf{n}_{ij}}{r_{ij}^2} \times \vec{\mu}_0 S_j^z \left(t - \frac{r_{ij}}{c} \right) . \quad (60)$$

The radiation magnetic field becomes

$$\mathbf{H}_{rad} \equiv \nabla \times \mathbf{A}_{rad} = \mathbf{H}_+ + \mathbf{H}_- + \mathbf{H}_z , \quad (61)$$

with

$$\begin{aligned} \mathbf{H}_+(\mathbf{r}_i, t) = & - \sum_{j(\neq i)}^N \left[\frac{\vec{\mu} - (\vec{\mu} \cdot \mathbf{n}_{ij})\mathbf{n}_{ij}}{c^2 r_{ij}} \frac{\partial^2}{\partial t^2} + \right. \\ & \left. + \frac{\vec{\mu} - 3(\vec{\mu} \cdot \mathbf{n}_{ij})\mathbf{n}_{ij}}{r_{ij}^3} \left(1 + \frac{r_{ij}}{c} \frac{\partial}{\partial t} \right) \right] S_j^+ \left(t - \frac{r_{ij}}{c} \right) \end{aligned} \quad (62)$$

and

$$\mathbf{H}_z(\mathbf{r}_i, t) = - \sum_{j(\neq i)}^N \frac{\vec{\mu}_0 - 3(\vec{\mu}_0 \cdot \mathbf{n}_{ij})\mathbf{n}_{ij}}{r_{ij}^3} S_j^z \left(t - \frac{r_{ij}}{c} \right) . \quad (63)$$

Let us define the effective radiation field

$$\mathbf{H}_{eff} \equiv \frac{1}{4\pi} \int \mathbf{H}_{rad} d\Omega(\mathbf{n}) \quad (64)$$

as field (61) averaged over spherical angles. In this averaging, we take into account the properties

$$\begin{aligned} \frac{1}{4\pi} \int [\vec{\mu} - (\vec{\mu} \cdot \mathbf{n})\mathbf{n}] d\Omega(\mathbf{n}) &= \frac{2}{3} \vec{\mu} , \\ \frac{1}{4\pi} \int [\vec{\mu} - 3(\vec{\mu} \cdot \mathbf{n})\mathbf{n}] d\Omega(\mathbf{n}) &= 0 , \quad \frac{1}{4\pi} \int (\vec{\mu} \cdot \mathbf{n})\mathbf{n} d\Omega(\mathbf{n}) = \frac{1}{3} \vec{\mu} . \end{aligned}$$

This gives

$$\frac{1}{4\pi} \int \mathbf{H}_z d\Omega(\mathbf{n}) = 0 . \quad (65)$$

As a result, we find

$$\mathbf{H}_{eff} = \mathbf{H}_{rad}^+ + \mathbf{H}_{rad}^- , \quad (66)$$

with

$$\mathbf{H}_{rad}^+(\mathbf{r}_i, t) = - \frac{2}{3} \vec{\mu} \sum_{j(\neq i)}^N \frac{1}{c^2 r_{ij}} \frac{\partial^2}{\partial t^2} S_j^+ \left(t - \frac{r_{ij}}{c} \right) . \quad (67)$$

Similarly to equations (22), we assume that

$$\frac{|\vec{\mu}_0 \cdot \mathbf{B}|}{\omega_0} \ll 1 , \quad \frac{|\vec{\mu} \cdot \mathbf{B}|}{\omega_0} \ll 1 , \quad (68)$$

hence the Born approximation can be invoked giving, similarly to section 2,

$$\mathbf{H}_{rad}^+(\mathbf{r}_i, t) = \frac{2}{3} k_0^3 \vec{\mu} \sum_{j(\neq i)}^N G^*(\mathbf{r}_{ij}, t) S_j^+(t) , \quad (69)$$

with the transfer function G introduced in section 4.

The radiation field (61) can be represented as a sum

$$\mathbf{H}_{rad} \equiv \mathbf{H}_{eff} + \mathbf{H}_{dip} \quad (70)$$

of the effective field (66) and the remaining part, for which

$$\int \mathbf{H}_{dip} d\Omega(\mathbf{n}) = 0 .$$

As in section 4, we introduce the effective forces acting on the atoms, caused by the external field,

$$f_0(\mathbf{r}, t) \equiv -2i\vec{\mu} \cdot \mathbf{H}_0(\mathbf{r}, t) , \quad (71)$$

due to radiation,

$$f_{rad}(\mathbf{r}, t) \equiv -2i\langle \vec{\mu} \cdot \mathbf{H}_{eff}(\mathbf{r}, t) \rangle , \quad (72)$$

and corresponding to local fluctuations

$$\xi \equiv -2i\vec{\mu} \cdot (\mathbf{H}_{vac} + \mathbf{H}_{dip} + \mathbf{H}_{mat}) . \quad (73)$$

Defining the natural linewidth

$$\gamma_0 \equiv \frac{2}{3} |\vec{\mu}|^2 k_0^3 , \quad (74)$$

we obtain the radiation force

$$f_{rad}(\mathbf{r}, t) = -i\gamma_0\rho \int [G(\mathbf{r} - \mathbf{r}', t)u(\mathbf{r}', t) + \mathbf{e}_\mu^2 G^*(\mathbf{r} - \mathbf{r}', t)u^*(\mathbf{r}', t)] d\mathbf{r}' , \quad (75)$$

in which $\mathbf{e}_\mu \equiv \vec{\mu}/|\vec{\mu}|$.

This force (75) enjoys the same form as that in equation (41) due to electric dipole transitions. The following analysis can be done in complete analogy with the case of electric dipole transitions, just replacing the electric transition dipole \mathbf{d} by the magnetic dipole $\vec{\mu}$.

7 Coherent and incoherent radiation

Some time ago, there has been a discussion in literature on the role of external fields in their possibility to influence the intensity of radiation. In particular, this problem has arisen in the study of gamma radiation of Mössbauer nuclei inside magnetic materials with large magnetic moments [17]. We shall investigate this problem in the next section, but meanwhile, we need to recall the general form of the radiation intensity to be studied later.

Below, we keep in mind electric dipole transitions, since, as has been explained above, mathematics for both, electric as well as magnetic dipoles, is the same. The local radiation intensity is defined as

$$I(\mathbf{n}, t) = \langle : \mathbf{n} \cdot \mathbf{S}(\mathbf{r}, t) : \rangle r^2 , \quad (76)$$

where the columns denote normal ordering, $\mathbf{n} \equiv \mathbf{r}/r$, and the Poynting vector is

$$\mathbf{S} \equiv \frac{c}{8\pi} (\mathbf{E}_{rad} \times \mathbf{H}_{rad} - \mathbf{H}_{rad} \times \mathbf{E}_{rad}) . \quad (77)$$

The radiation fields are given by the expressions

$$\mathbf{E}_{rad} = -\frac{1}{c} \frac{\partial \mathbf{A}_{rad}}{\partial t} , \quad \mathbf{H}_{rad} = \nabla \times \mathbf{A}_{rad} . \quad (78)$$

In the wave zone, one uses the approximation

$$|\mathbf{r} - \mathbf{r}_j| \simeq r - \mathbf{n} \cdot \mathbf{r}_j \quad (r \gg |\mathbf{r}_j|) . \quad (79)$$

Then the radiation vector potential can be written as

$$\mathbf{A}_{rad} = \mathbf{A}_+ + \mathbf{A}_- , \quad (80)$$

with

$$\mathbf{A}_+(\mathbf{r}, t) \simeq i \frac{k_0}{r} \mathbf{d} \sum_{j=1}^N S_j^+ \left(t - \frac{r - \mathbf{n} \cdot \mathbf{r}_j}{c} \right) \quad (81)$$

and \mathbf{A}_- being Hermitian conjugate to \mathbf{A}_+ . Using this, we get the electric radiation field

$$\mathbf{E}_{rad} \simeq -ik_0(\mathbf{A}_+ - \mathbf{A}_-) \quad (82)$$

and magnetic radiation field

$$\mathbf{H}_{rad} \simeq \mathbf{n} \times \mathbf{E}_{rad} . \quad (83)$$

The radiation intensity (76) becomes

$$I(\mathbf{n}, t) = \frac{cr^2}{4\pi} \langle : \mathbf{E}_{rad}^2 - (\mathbf{n} \cdot \mathbf{E}_{rad})^2 : \rangle , \quad (84)$$

which, treating the retardation in the Born approximation, is reduced to

$$I(\mathbf{n}, t) = 2\omega_0\gamma_0 \sum_{ij}^N \varphi_{ij}(\mathbf{n}) \langle S_i^+(t) S_j^-(t) \rangle , \quad (85)$$

where the formfactor

$$\varphi_{ij}(\mathbf{n}) \equiv \frac{3}{8\pi} |\mathbf{n} \times \mathbf{e}_d|^2 \exp(ik_0 \mathbf{n} \cdot \mathbf{r}_{ij}) \quad (86)$$

is introduced.

The radiation intensity averaged over random fluctuations and the period of fast oscillations

$$\bar{I}(\mathbf{n}, t) \equiv \frac{\omega_0}{2\pi} \int_t^{t+2\pi/\omega_0} \langle \langle I(\mathbf{n}, t') \rangle \rangle dt' \quad (87)$$

can be presented as a sum

$$\bar{I}(\mathbf{n}, t) = I_{inc}(\mathbf{n}, t) + I_{coh}(\mathbf{n}, t) \quad (88)$$

of the incoherent radiation intensity

$$I_{inc}(\mathbf{n}, t) = \omega_0\gamma_0 \sum_{j=1}^N \varphi_{jj}(\mathbf{n}) [1 + s_j(t)] \quad (89)$$

and the coherent radiation intensity

$$I_{coh}(\mathbf{n}, t) = \frac{1}{2} \omega_0\gamma_0 \sum_{i \neq j}^N \varphi_{ij}(\mathbf{n}) \overline{u_i^*(t) u_j(t)} . \quad (90)$$

The diagonal formfactor is

$$\varphi_{jj}(\mathbf{n}) = \frac{3}{8\pi} |\mathbf{n} \times \mathbf{e}_d|^2 \equiv \varphi(\mathbf{n}) . \quad (91)$$

The total radiation intensity, integrated over the spherical angles,

$$I(t) \equiv \int \bar{I}(\mathbf{n}, t) d\Omega(\mathbf{n}) = I_{inc}(t) + I_{coh}(t) , \quad (92)$$

consists of the incoherent part

$$I_{inc}(t) = \omega_0 \gamma_0 \sum_{j=1}^N [1 + s_j(t)] \quad (93)$$

and the coherent part

$$I_{coh}(t) = \frac{1}{2} \omega_0 \gamma_0 \sum_{i \neq j}^N \overline{\varphi_{ij} u_i^*(t) u_j(t)} . \quad (94)$$

Here the shape factor is defined as

$$\varphi_{ij} \equiv \int \varphi_{ij}(\mathbf{n}) d\Omega(\mathbf{n}) , \quad \varphi_{jj} \equiv \int \varphi(\mathbf{n}) d\Omega(\mathbf{n}) = 1 . \quad (95)$$

In this way, we see that to calculate the radiation intensity, we need to find the solutions to the evolution equations (43), (44), and (45).

8 Influence of external fields

Now we shall study how external fields influence the radiation intensity. We are interested in the permanent action of external fields, because of which we shall not pay attention to short temporary coherent effects. Then the evolution equations (45) can be represented as

$$\begin{aligned} \frac{\partial u}{\partial t} &= -(i\omega_0 + \gamma_2)u - 2is\mathbf{d} \cdot \mathbf{E}_0(\mathbf{r}, t) , \\ \frac{\partial w}{\partial t} &= -2\gamma_2 w - 2is(u^*\mathbf{d} - u\mathbf{d}^*) \cdot \mathbf{E}_0(\mathbf{r}, t) , \\ \frac{\partial s}{\partial t} &= i(u^*\mathbf{d} - u\mathbf{d}^*) \cdot \mathbf{E}_0(\mathbf{r}, t) - \gamma_1(s - \zeta) . \end{aligned} \quad (96)$$

The external field

$$\mathbf{E}_0(\mathbf{r}, t) = \mathbf{E}_0 + \mathbf{E}_1 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \mathbf{E}_1^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \quad (97)$$

consists of a constant and an alternating field. As usual, there are the following small parameters

$$\frac{\gamma_1}{\omega_0} \ll 1 , \quad \frac{\gamma_2}{\omega_0} \ll 1 , \quad \frac{|\mathbf{d} \cdot \mathbf{E}_0|}{\omega_0} \ll 1 , \quad \frac{|\mathbf{d} \cdot \mathbf{E}_1|}{\omega_0} \ll 1 . \quad (98)$$

The alternating field is tuned to resonance, so that the resonance condition

$$\frac{|\Delta|}{\omega_0} \ll 1 \quad (\Delta \equiv \omega - \omega_0) \quad (99)$$

is valid.

The problem can be solved in the scale separation approach, as explained in section 5. According to the existing small parameters, the variable u is treated as fast, while w and s , as slow. For the fast variable, we get

$$u = u_0 e^{-(i\omega_0 + \gamma_2)t} - 2is\mathbf{d} \cdot \int_0^t \mathbf{E}_0(\mathbf{r}, t - t') e^{-(i\omega_0 + \gamma_2)t'} dt' , \quad (100)$$

where u_0 is an initial value of u . In view of field (97), we find

$$\begin{aligned} u = & -\frac{2s\mathbf{d} \cdot \mathbf{E}_0}{\omega_0 - i\gamma_2} + \frac{2s\mathbf{d} \cdot \mathbf{E}_1}{\Delta + i\gamma_2} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \\ & + \left(u_0 + \frac{2s\mathbf{d} \cdot \mathbf{E}_0}{\omega_0 - i\gamma_2} - \frac{2s\mathbf{d} \cdot \mathbf{E}_1}{\Delta + i\gamma_2} e^{i\mathbf{k} \cdot \mathbf{r}} \right) e^{-(i\omega_0 + \gamma_2)t} . \end{aligned} \quad (101)$$

For the guiding center of the slow variable s , we obtain the equation

$$\frac{\partial s}{\partial t} = -\gamma_1^* (s - \zeta^*) , \quad (102)$$

in which

$$\gamma_1^* = \gamma_1 + 16\gamma_2 \left(\frac{|\mathbf{d} \cdot \mathbf{E}_0|^2}{\omega_0^2 + \gamma_2^2} + \frac{|\mathbf{d} \cdot \mathbf{E}_1|^2}{\Delta^2 + \gamma_2^2} \right) , \quad \zeta^* = \frac{\gamma_1}{\gamma_1^*} \zeta . \quad (103)$$

The solution to equation (102) is

$$s = s_0 e^{-\gamma_1^* t} + (1 - e^{-\gamma_1^* t}) \zeta^* . \quad (104)$$

Here s_0 is an initial value of s . For simplicity, we accept a uniform initial condition $s_0(\mathbf{r}) \equiv s(\mathbf{r}, 0) = s_0$. Then solution (104) is also uniform, $s(\mathbf{r}, t) = s(t)$.

Averaging over fast oscillations, we have

$$\overline{u_i^* u_j} = 4s_i s_j \left(\frac{|\mathbf{d} \cdot \mathbf{E}_0|^2}{\omega_0^2 + \gamma_2^2} + \frac{|\mathbf{d} \cdot \mathbf{E}_1|^2}{\Delta^2 + \gamma_2^2} e^{-i\mathbf{k} \cdot \mathbf{r}_{ij}} \right) ,$$

which should be substituted into the radiation intensity (94). The incoherent radiation intensity is

$$I_{inc}(\mathbf{n}, t) = N\omega_0\gamma_0\varphi(\mathbf{n})[1 + s(t)] . \quad (105)$$

For the coherent radiation intensity, we find

$$I_{coh}(\mathbf{n}, t) = 2N^2\omega_0\gamma_0\varphi(\mathbf{n})s^2(t) \left[F(k_0\mathbf{n}) \frac{|\mathbf{d} \cdot \mathbf{E}_0|^2}{\omega_0^2 + \gamma_2^2} + F(k_0\mathbf{n} - \mathbf{k}) \frac{|\mathbf{d} \cdot \mathbf{E}_1|^2}{\Delta^2 + \gamma_2^2} \right] , \quad (106)$$

with the formfactor

$$F(\mathbf{k}) \equiv \left| \frac{1}{N} \sum_{j=1}^N e^{i\mathbf{k} \cdot \mathbf{r}_j} \right|^2 . \quad (107)$$

In the case of a cylindrical sample of radius R and length L , with $\mathbf{k} = k_0\mathbf{e}_z$, the formfactors in equation (106) read

$$F(k_0\mathbf{n}) = \frac{\lambda^4}{\pi^4 R^2 L^2 \sin^2 \vartheta \cos^2 \vartheta} J_1^2 \left(\frac{2\pi R}{\lambda} \sin \vartheta \right) \sin^2 \left(\frac{\pi L}{\lambda} \cos \vartheta \right) ,$$

$$F(k_0 \mathbf{n} - \mathbf{k}) = \frac{\lambda^4}{\pi^4 R^2 L^2 \sin^2 \vartheta (1 - \cos \vartheta)^2} J_1^2 \left(\frac{2\pi R}{\lambda} \sin \vartheta \right) \sin^2 \left[\frac{\pi L}{\lambda} (1 - \cos \vartheta) \right] ,$$

where $\mathbf{k} = k_0 \mathbf{e}_z$, $k_0 = 2\pi/\lambda$, and ϑ is the angle between \mathbf{n} and the axis z .

The maximal value of the formfactor is one. For the formfactor $F(k_0 \mathbf{n})$, this happens under the condition

$$k_0 \mathbf{n} \cdot \mathbf{r}_j = 2\pi n_j \quad (n_j = 0, \pm 1, \pm 2, \dots) . \quad (108)$$

For instance, if $n_j = 0$, condition (108) is valid for a chain of atoms with \mathbf{r}_j perpendicular to the direction of \mathbf{n} .

The factor $F(k_0 \mathbf{n} - \mathbf{k})$ is maximal, reaching one, when

$$(k_0 \mathbf{n} - \mathbf{k}) \cdot \mathbf{r}_j = 2\pi n_j \quad (n_j = 0, \pm 1, \pm 2, \dots) . \quad (109)$$

In the present case, we consider radiating atoms. But similar conditions take place for the case of an atomic system scattering external radiation. Thus, the case $n_j = 0$ corresponds to forward or backward scattering. When $n_j \neq 0$, the atoms have to form an ideal lattice, and equation (109) is the condition of the Bragg scattering. The increase of the scattering intensity under a lattice arrangement of atoms is called the Borrmann effect [18, 19].

The total radiation intensity (92) is the sum of the incoherent intensity

$$I_{inc}(t) = N\omega_0\gamma_0[1 + s(t)] \quad (110)$$

and the coherent radiation intensity

$$I_{coh}(t) = 2N^2\omega_0\gamma_0 s^2(t) \left(\frac{\varphi_0 |\mathbf{d} \cdot \mathbf{E}_0|^2}{\omega_0^2 + \gamma_2^2} + \frac{\varphi_1 |\mathbf{d} \cdot \mathbf{E}_1|^2}{\Delta^2 + \gamma_2^2} \right) . \quad (111)$$

In the latter, the notation for the shape factors

$$\begin{aligned} \varphi_0 &\equiv \int \varphi(\mathbf{n}) F(k_0 \mathbf{n}) d\Omega(\mathbf{n}) , \\ \varphi_1 &\equiv \int \varphi(\mathbf{n}) F(k_0 \mathbf{n} - \mathbf{k}) d\Omega(\mathbf{n}) \end{aligned} \quad (112)$$

are used.

The values of the shape factors essentially depend on the shape of the radiating sample and on the type of the dipole transition characterized by the change of the quantum number Δm . For instance

$$\mathbf{e}_d = \begin{cases} \mathbf{e}_z & (\Delta m = 0) \\ \frac{1}{\sqrt{2}} (\mathbf{e}_x \mp i\mathbf{e}_y) & (\Delta m = \pm 1) \end{cases} .$$

Therefore,

$$|\mathbf{n} \times \mathbf{e}_d|^2 = \begin{cases} 1 - \cos^2 \vartheta & (\Delta m = 0) \\ 1 - \frac{1}{2} \sin^2 \vartheta & (\Delta m = \pm 1) \end{cases} .$$

In particular, for a pencil-like sample or a disk-like sample and $\Delta m = \pm 1$, one has [1], respectively,

$$\varphi_1 \simeq \begin{cases} \frac{3\lambda}{8L} & (\frac{\lambda}{2\pi L} \ll 1, \frac{R}{L} \ll 1) \\ \frac{3}{8} \left(\frac{\lambda}{\pi R} \right)^2 & (\frac{\lambda}{2\pi R} \ll 1, \frac{L}{R} \ll 1) \end{cases} . \quad (113)$$

As follows from these results, external fields increase the longitudinal attenuation, which leads to the accelerated relaxation of s . They also induce coherent radiation, but weakly influence incoherent radiation.

The alternating resonant external field usually produces a stronger effect than the constant field. The latter can play a more important role than the alternating field, if

$$\varphi_1 \frac{|\mathbf{d} \cdot \mathbf{E}_1|}{\gamma_2} < \varphi_0 \frac{|\mathbf{d} \cdot \mathbf{E}_0|}{\omega_0} \ll 1. \quad (114)$$

In other words, when

$$\frac{\varphi_0 \gamma_2 |\mathbf{E}_0|}{\varphi_1 \omega_0 |\mathbf{E}_1|} > 1. \quad (115)$$

The problem of possible influence of a constant external field on the coherent radiation intensity was discussed [7, 8, 20, 21] with respect to the occurrence of the so-called Mössbauer magnetic anomaly. The latter consists in the increase of the spectrum area, when paramagnetic state changes to ferromagnetic state [17, 22]. In some papers, such an increase was associated with the influence of the external field on the phonon characteristics of the material. This interpretation, however, was shown to be incorrect [8].

In the case of the Mössbauer radiation by ^{57}Fe , the typical, parameters are $\omega_0 \sim 10^{19}$ 1/s, $\gamma_2 \sim 10^7$ 1/s, $\lambda \sim 10^{-8}$ cm, $H_0 \sim 10^5$ G, and $H_1 \sim 10^{-5}$ G. This gives

$$\frac{\gamma_2 |\mathbf{H}_0|}{\omega_0 |\mathbf{H}_1|} \sim 10^{-2}.$$

If φ_0 and φ_1 are of the same order of magnitude, then the appearing constant magnetic field should not play role in such typical conditions. It could play role for a very weak alternating field of order $H_1 \sim 10^{-7}$ G. The magnetic anomaly in ferromagnets could be due to the inhomogeneous broadening effect [8].

9 Triggering dipolar waves

If an atomic system is prepared in an excited state, then it starts radiating by spontaneous emission that leads to the arising correlations between atoms through the radiated field. The semiclassical approximation cannot describe this process. This approximation requires that an initial coherence be imposed upon the system, hence it is applicable only to the coherent stage of radiation.

In the frame of the pseudospin representation, the process of radiation starts with spontaneous atomic radiation, resulting in the self-action, and the appearing atomic correlations are associated with triggering dipolar waves [23]. This initial stage of the atomic state can be described by the pseudospin equations

$$\begin{aligned} \frac{dS_j^-}{dt} &= -i\omega_0 S_j^- - iS_j^z \sum_{i(\neq j)}^N (b_{ji} S_i^+ - c_{ji} S_i^-), \\ \frac{dS_j^z}{dt} &= \frac{i}{2} \sum_{i(\neq j)}^N [S_j^+ (b_{ji} S_i^+ - c_{ji} S_i^-) - S_j^- (b_{ji}^* S_i^- - c_{ji}^* S_i^+)], \end{aligned} \quad (116)$$

in which

$$b_{ij} \equiv \frac{k_0^2}{2\pi} \sum_{\alpha\beta} d^\alpha D_{ij}^{\alpha\beta} d^\beta , \quad c_{ij} \equiv \frac{k_0^2}{2\pi} \sum_{\alpha\beta} d^\alpha \left(D_{ij}^{\alpha\beta} d^\beta \right)^* ,$$

$$D_{ij}^{\alpha\beta} \equiv \int \Theta(ct - |\mathbf{r}_i - \mathbf{r}|) \frac{D_{\alpha\beta}(\mathbf{r} - \mathbf{r}_j)}{|\mathbf{r}_j - \mathbf{r}|} \exp(-ik_0|\mathbf{r}_i - \mathbf{r}|) d\mathbf{r} .$$

Triggering dipolar waves correspond to small deviations from the average pseudospin values:

$$S_j^\alpha = \langle S_j^\alpha \rangle + \delta S_j^\alpha . \quad (117)$$

To zero order, we have the equations

$$\frac{d}{dt} \langle S_j^- \rangle = -i\omega_0 \langle S_j^- \rangle , \quad \frac{d}{dt} \langle S_j^z \rangle = 0 , \quad (118)$$

whose solutions are

$$\langle S_j^-(t) \rangle = \langle S_j^-(0) \rangle e^{-i\omega_0 t} , \quad \langle S_j^z(t) \rangle = \langle S_j^z(0) \rangle . \quad (119)$$

The equations for small deviations are

$$\frac{d}{dt} \delta S_j^- = -i\omega_0 \delta S_j^- - i \langle S_j^z \rangle \sum_{i(\neq j)}^N (b_{ji} \delta S_i^+ - c_{ji} \delta S_i^-) ,$$

$$\frac{d}{dt} \delta S_j^z = \frac{i}{2} \sum_{i(\neq j)}^N [(b_{ji} \delta S_i^+ - c_{ji} \delta S_i^-) \langle S_j^+ \rangle - (b_{ji}^* \delta S_i^- - c_{ji}^* \delta S_i^+) \langle S_j^- \rangle] . \quad (120)$$

Substituting into equations (120) the Fourier transforms for the deviations,

$$\delta S_j^- = \sum_k \delta S_k^- e^{i\mathbf{k} \cdot \mathbf{r}_j} , \quad \delta S_j^+ = \sum_k \delta S_k^+ e^{-i\mathbf{k} \cdot \mathbf{r}_j} , \quad (121)$$

and for the coefficients

$$b_{ij} = \frac{1}{N} \sum_k b_k e^{i\mathbf{k} \cdot \mathbf{r}_{ij}} , \quad c_{ij} = \frac{1}{N} \sum_k c_k e^{i\mathbf{k} \cdot \mathbf{r}_{ij}} ,$$

we obtain the equations

$$\frac{d}{dt} \delta S_k^- = -i\mu_k \delta S_k^- - i\lambda_k \delta S_k^+ ,$$

$$\frac{d}{dt} \delta S_k^+ = i\mu_k^* \delta S_k^+ + i\lambda_k^* \delta S_k^- , \quad (122)$$

in which

$$\mu_k \equiv \omega_0 - c_k \langle S_j^z \rangle , \quad \lambda_k \equiv b_{-k} \langle S_j^z \rangle .$$

The solutions to equations (122) can be represented in the form

$$\delta S_k^- = u_k e^{-i\omega_k t} + v_k^* e^{i\omega_k t} , \quad \delta S_k^+ = u_k^* e^{i\omega_k t} + v_k e^{-i\omega_k t} , \quad (123)$$

with the dipolar wave spectrum

$$\omega_k = \sqrt{|\mu_k|^2 - |\lambda_k|^2} . \quad (124)$$

Because of the inequalities

$$\frac{|b_k|}{\omega_0} \ll 1, \quad \frac{|c_k|}{\omega_0} \ll 1, \quad \left| \frac{\lambda_k}{\mu_k} \right| \ll 1,$$

the spectrum is positive, so that the dipolar waves are stable. In the long-wave limit, the spectrum reads

$$\omega_k \simeq \omega_0 + \frac{1}{2} \langle S_i^z \rangle \sum_{j(\neq i)}^N (\Re c_{ij}) (\mathbf{k} \cdot \mathbf{r}_{ij})^2. \quad (125)$$

The dipolar waves trigger the process of self-organization in a radiating atomic system.

10 Transverse mode expansion

When the radiation wavelength λ is much shorter than the characteristic sizes of the atomic sample, the radiating beam cannot be uniform, but separates into filaments [23]. The details of such a filamentation will be treated in a following section. Meanwhile, we just accept the possible existence of such filaments and describe the general way of treating them. We consider the situation, when the radiation propagates along the axis z as a plane wave with the seed frequency $\omega = ck$.

Let the sample be of cylindrical shape, with radius R and length L which are much larger than the radiation wavelength,

$$\frac{\lambda}{R} \ll 1, \quad \frac{\lambda}{L} \ll 1. \quad (126)$$

Suppose that there are N_f elementary modes in the sample. Each filament can be surrounded by an enveloping cylinder of volume $V_f = \pi R_f^2 L$. The separation of the radiating beam into filaments implies that the solutions to the evolution equations (43) to (45) can be represented as the expansions

$$\begin{aligned} u(\mathbf{r}, t) &= \sum_{n=1}^{N_f} u_n(r_\perp, t) e^{ikz}, & w(\mathbf{r}, t) &= \sum_{n=1}^{N_f} w_n(r_\perp, t), \\ s(\mathbf{r}, t) &= \sum_{n=1}^{N_f} s_n(r_\perp, t), \end{aligned} \quad (127)$$

over the transverse modes, where $r_\perp \equiv \sqrt{x^2 + y^2}$ is the transverse radial variable. For each filament, one can define the averaged solutions

$$\begin{aligned} u(t) &\equiv \frac{1}{V_f} \int u_n(r_\perp, t) d\mathbf{r} = \frac{2}{R_f^2} \int_0^{R_f} u_n(r, t) r dr, \\ w(t) &\equiv \frac{1}{V_f} \int w_n(r_\perp, t) d\mathbf{r} = \frac{2}{R_f^2} \int_0^{R_f} w_n(r, t) r dr, \\ s(t) &\equiv \frac{1}{V_f} \int s_n(r_\perp, t) d\mathbf{r} = \frac{2}{R_f^2} \int_0^{R_f} s_n(r, t) r dr, \end{aligned} \quad (128)$$

averaged over the related enveloping volumes.

We introduce the effective coupling functions

$$\alpha(t) \equiv \gamma_0 \rho \int \Theta(ct - r) \frac{\sin(k_0 r - kz)}{k_0 r} d\mathbf{r} , \quad (129)$$

and

$$\beta(t) \equiv \gamma_0 \rho \int \Theta(ct - r) \frac{\cos(k_0 r - kz)}{k_0 r} d\mathbf{r} . \quad (130)$$

Also, we define the average stochastic variable

$$\xi(t) \equiv \frac{1}{V_f} \int \xi(\mathbf{r}, t) e^{-ikz} d\mathbf{r} . \quad (131)$$

Employing the scale separation approach, we meet the nonresonant terms of the type

$$\begin{aligned} & (\alpha + i\beta) s u^* \mathbf{e}_d^2 , \\ & s(\alpha + i\beta) (u^* \mathbf{e}_d)^2 + s(\alpha - i\beta) (\mathbf{e}_d^* u)^2 , \\ & \frac{1}{2} (\alpha + i\beta) (u^* \mathbf{e}_d)^2 + \frac{1}{2} (\alpha - i\beta) (\mathbf{e}_d^* u)^2 , \end{aligned}$$

which give the contribution of order γ_0/ω_0 , as compared to resonant terms, because of which such terms can be safely neglected.

For the averaged solutions (128), we obtain the equations

$$\begin{aligned} \frac{du}{dt} &= -i(\omega_0 + \beta s)u - (\gamma_2 - \alpha s)u + \xi s , \\ \frac{dw}{dt} &= -2(\gamma_2 - \alpha s)w + (u^* \xi + \xi^* u) s , \\ \frac{ds}{dt} &= -\alpha w - \frac{1}{2} (u^* \xi + \xi^* u) s - \gamma_1 (s - \zeta) . \end{aligned} \quad (132)$$

We introduce the effective attenuation

$$\Gamma \equiv \gamma_2 - \alpha s , \quad (133)$$

taking into account collective processes, and the effective frequency

$$\Omega \equiv \omega_0 + \beta s , \quad (134)$$

including the collective Lamb shift. The solution for the fast variable takes the form

$$u = u_0 e^{-(i\Omega + \Gamma)t} + s \int_0^t \xi(t - t') e^{-(i\Omega + \Gamma)t'} dt' . \quad (135)$$

The stochastic variable (131) is assumed to be zero centered,

$$\langle \langle \xi(t) \rangle \rangle = 0 . \quad (136)$$

Also, we shall need the quantity

$$\gamma_3 \equiv \text{Re} \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau dt \int_0^t \langle \langle \xi^*(t) \xi(t - t') \rangle \rangle e^{-(i\Omega + \Gamma)t'} dt' , \quad (137)$$

playing the role of a dynamic attenuation caused by the stochastic variables. Actually, it is admissible to set the correlation property

$$\langle\langle \xi^*(t)\xi(t') \rangle\rangle = 2\gamma_3\delta(t-t') \quad (138)$$

that reduces definition (137) to an identity.

Following the scale separation approach, we substitute the fast variable (135) into the equations for slow variables and average the latter over time and over the stochastic variable ξ . This gives the equation for the coherence intensity

$$\frac{dw}{dt} = -2(\gamma_2 - \alpha s)w + 2\gamma_3 s^2, \quad (139)$$

and for the population imbalance

$$\frac{ds}{dt} = -\alpha w - \gamma_3 s - \gamma_1(s - \zeta). \quad (140)$$

These equations are applicable to all stages of atomic radiation.

11 Emergence of coherence from chaos

An excited atomic system passes through several qualitatively different dynamic stages, similarly to the relaxation of any statistical system from a nonequilibrium state [24]. The first is the interaction stage,

$$0 < t < t_{int} \quad (\text{interaction stage}), \quad (141)$$

during which atoms begin spontaneous radiation, but radiate independently from each other, having yet no time for developing mutual interactions. The initial values of the coherence intensity and population imbalance practically do not change,

$$w(t_{int}) \approx w_0, \quad s(t_{int}) \approx s_0. \quad (142)$$

The interaction time is very short, being of order $t_{int} \sim a/c$, where a is the mean interatomic distance.

After the interaction time, atoms start feeling each other through the photon exchange, but there is no yet correlation between them, so that they radiate rather independently from each other. This is the chaotic quantum stage, lasting in the interval

$$t_{int} < t < t_{coh} \quad (\text{chaotic stage}), \quad (143)$$

before the coherence time, when strong correlations between atoms appear. After the interaction time, the effective coupling functions (129) and (130) grow as

$$\alpha(t) \rightarrow g\gamma_2, \quad \beta(t) \rightarrow g'\gamma_2 \quad (t > t_{int}),$$

where the dimensionless coupling parameters are

$$g \equiv \rho \frac{\gamma_0}{\gamma_2} \int \frac{\sin(k_0 r - kz)}{k_0 r} d\mathbf{r} \quad (144)$$

and

$$g' \equiv \rho \frac{\gamma_0}{\gamma_2} \int \frac{\cos(k_0 r - kz)}{k_0 r} d\mathbf{r} . \quad (145)$$

The integration here is over V_f . If no initial coherence is imposed on the excited system, so that

$$w_0 \equiv w(0) = 0 , \quad s_0 \neq 0 , \quad (146)$$

then the evolution equations are

$$\frac{dw}{dt} = 2\gamma_3 s^2 , \quad \frac{ds}{dt} = -(\gamma_1 + \gamma_3)s + \gamma_1 \zeta . \quad (147)$$

The coherence time satisfies the inequality

$$(\gamma_1 + \gamma_3)t_{coh} \ll 1 . \quad (148)$$

At this chaotic stage, the solutions to equations (147) are

$$w \simeq 2\gamma_3 s_0^2 t , \quad s \simeq s_0 - [(\gamma_1 + \gamma_3)s_0 - \gamma_1 \zeta]t . \quad (149)$$

The coherence time corresponds to the point where the collective term in equation (139) becomes comparable with the chaotic term due to quantum fluctuations, that is, when

$$\gamma_2(gs - 1)w = \gamma_3 s^2 \quad (t = t_{coh}) . \quad (150)$$

This may happen under a sufficient initial atomic excitation and a strong coupling, when $gs_0 > 1$. Then the coherence time is

$$t_{coh} = \frac{s_0}{2[\gamma_2(gs_0 - 1)s_0 + \gamma_3 s_0 + \gamma_1(s_0 - \zeta)]} . \quad (151)$$

If the coupling parameter is large, then

$$t_{coh} \simeq \frac{T_2}{2gs_0} \quad (gs_0 \gg 1) , \quad (152)$$

where $T_2 \equiv \gamma_2^{-1}$. We may notice that the system can reach the coherence time only if it is initially excited, so that $s_0 > 0$. At the end of the chaotic stage, the solutions are

$$w(t_{coh}) \simeq 2\gamma_3 s_0^2 t_{coh} , \quad s(t_{coh}) \simeq s_0 . \quad (153)$$

As soon as coherence develops in the system, the coherent stage comes into play, exhibiting superradiance occurring in the interval

$$t_{coh} < t < T_2 \quad (coherent \ stage) . \quad (154)$$

At this stage, collective effects are dominant, so that $\gamma_1 \ll g\gamma_2$ and $\gamma_3 \ll g\gamma_2$. The dynamics is described by the equations

$$\frac{dw}{dt} = -2\gamma_2(1 - gs)w , \quad \frac{ds}{dt} = -g\gamma_2 w , \quad (155)$$

whose solutions are

$$w = \left(\frac{\gamma_p}{g\gamma_2} \right)^2 \text{sech}^2 \left(\frac{t - t_0}{\tau_p} \right) , \quad s = \frac{1}{g} - \frac{\gamma_p}{g\gamma_2} \tanh \left(\frac{t - t_0}{\tau_p} \right) . \quad (156)$$

Here the notations are used:

$$\gamma_p^2 = \gamma_g^2 + 2g^2\gamma_2^2\gamma_3s_0^2t_{coh} , \quad \gamma_g = (gs_0 - 1)\gamma_2 . \quad (157)$$

The delay time, where the coherence intensity is maximal, reads

$$t_0 = t_{coh} + \frac{\tau_p}{2} \ln \left| \frac{\gamma_p + \gamma_g}{\gamma_p - \gamma_g} \right| . \quad (158)$$

The superradiant pulse time is

$$\tau_p \equiv \frac{1}{\gamma_p} = \frac{T_2}{gs_0 - 1} \left[1 - \frac{g^2s_0^2\gamma_3t_{coh}}{(gs_0 - 1)^2} \right] . \quad (159)$$

If the atomic coupling is strong, such that $gs_0 \gg 1$, then

$$t_0 \simeq 5t_{coh} , \quad \tau_p \simeq 2t_{coh} ,$$

and the delay time becomes

$$t_0 = t_{coh} \left(1 + \ln \left| \frac{2}{\gamma_3t_{coh}} \right| \right) . \quad (160)$$

Coherence dies out at the scale of T_2 , after which the system relaxes during the relaxation stage

$$T_2 < t < T_1 \quad (\text{relaxation stage}) , \quad (161)$$

with the decaying solutions

$$w \simeq \left(\frac{2\gamma_p}{g\gamma_2} \right)^2 \exp \left(-\frac{2t}{\tau_p} \right) , \quad s \simeq \frac{\gamma_2 - \gamma_p}{g\gamma_2} + \frac{2\gamma_p}{g\gamma_2} \exp \left(-\frac{2t}{\tau_p} \right) . \quad (162)$$

The longitudinal relaxation time is $T_1 = 1/\gamma_1$.

Finally, for large times

$$t > T_1 \quad (\text{quasi-stationary stage}) , \quad (163)$$

the quasi-stationary stage settles down, described by the equations

$$\frac{dw}{dt} = -2\gamma_2(1 - gs)w + 2\gamma_3s^2 , \quad \frac{ds}{dt} = -g\gamma_2w - \gamma_3s - \gamma_1(s - \zeta) . \quad (164)$$

Stationary solutions are defined by the zeroes of the right-hand sides of these equations, and small oscillations around the stationary solutions characterize the remnants of radiation [8].

In this way, when no initial coherence has been imposed upon the atomic system, its evolution passes through the following qualitatively different temporal intervals: interaction stage, chaotic stage, coherent stage, relaxation stage, and quasi-stationary stage. Coherent radiation during the coherent stage corresponds to pure superradiance. Here the term *pure* stresses that

superradiance develops as a self-organized spontaneous process, without being forced with external fields. The self-organization of atomic radiation, due to mutual correlations through the common radiation field, is the essence of the Dicke effect [25].

However, if at the initial moment of time, a coherent pulse is imposed upon the atomic system, then the chaotic stage can be essentially shortened, depending on the pulse intensity. This regime is called triggered superradiance. If the initial coherent pulse is strong, then the chaotic stage may disappear at all, so that after the interaction stage, the coherent stage comes into play. Then the superradiant pulse time becomes

$$\tau_p = \frac{1}{\sqrt{\gamma_g^2 + (g\gamma_2)^2 w_0}} . \quad (165)$$

The evolution after the coherent stage is the same as in the case of pure superradiance.

In all the cases, the transitions between different dynamic stages are not absolutely sharp, but are rather gradual crossovers.

12 Pulsing and punctuated superradiance

If the atomic system is subject to non-resonant pumping supporting the condition

$$g\zeta \gg 1 ,$$

then there appears a series of superradiant pulses, because of which this regime can be named *pulsing superradiance* [8]. After a series of superradiant bursts, the solutions tend to the stationary state

$$w^* \simeq \frac{\gamma_1 \zeta}{\gamma_2 g} , \quad s^* \simeq \frac{1}{g} \left(1 - \frac{\gamma_3}{\gamma_1 g \zeta} \right) ,$$

which is a stable focus. The intervals between the superradiant pulses approximately are given by the period

$$T_{eff} = \pi \sqrt{\frac{2T_1 T_2}{g\zeta}} .$$

Another superradiant regime can be organized by acting on the atomic system with π pulses inverting the atomic population at the chosen instants of time. Then the regime of *punctuated superradiance* occurs [24].

13 Turbulent photon filamentation

In nonlinear media interacting with electromagnetic fields there appear different spatiotemporal structures that are analogous to the structures arising in many other complex nonequilibrium systems [8, 26–30]. The most known among such electromagnetic structures are the optical filaments which can be formed in passive nonlinear matter [28–30] and in active laser media [8, 26, 29].

The behaviour and characteristics of optical filaments, arising in laser media, essentially depend on the value of the Fresnel number $F \equiv R^2/\lambda L$, in which R and L are the internal radius (aperture radius) and effective length, respectively, of a cylindrical laser sample, and

λ is the optical wavelength. There are two types of optical filaments, regular and turbulent, corresponding to either low or high Fresnel numbers.

Here we consider the turbulent photon filamentation. The theory of the turbulent filamentation in laser media was advanced, first, on the basis of stationary models [31–34] invoking the notion of an effective time-averaged energy. A more elaborate approach, based on realistic evolution equations, was developed later [8, 35–38].

As is mentioned above, there are two types of optical filaments, regular and turbulent, which is related to the value of the laser Fresnel number. The latter plays for optical systems the same role as the Reynolds number for moving fluids. When increasing the Reynolds number, a laminar fluid transforms into turbulent. In the similar manner, increasing the Fresnel number makes a regular filamentary structure turbulent. Optical turbulence implies, by analogy with the fluid turbulence, that the spatiotemporal dynamics is chaotic. This means that the radiating filaments are randomly distributed in space and are not correlated with each other.

In experiments, optical filaments are usually observed in the near-field cross-section of lasers. The typical picture, when varying the Fresnel number is as follows. At very small Fresnel numbers $F \ll 1$, there exists the sole transverse central mode uniformly filling the laser medium. When the Fresnel number is around $F \sim 1$, the laser cavity can house several transverse modes seen as a regular arrangement of bright spots in the transverse cross-section. Each mode corresponds to a filament extended through the cylindrical volume. This filamentary structure is regular in space, forming ordered geometric arrays, such as polygons. The transverse structure is imposed by the cavity geometry, being prescribed by the empty-cavity Gauss-Laguerre modes. Such regular structures are well understood theoretically; their description is based on the field expansion over the modal Gauss-Laguerre functions related to the cylindrical geometry [26]. For Fresnel numbers up to $F \approx 5$, the number of bright filaments follows the F^2 law as F increases. The regular filamentary structures have been observed in several lasers, such as CO_2 and Na_2 lasers [26]. Similar structures also appear in many passive nonlinear media, e.g. in Kerr medium and in active nonlinear media, as the photorefractive $\text{Bi}_{12}\text{SiO}_2$ crystal pumped by a laser [28, 29].

As soon as the Fresnel number reaches $F \approx 10$, there occurs a qualitative change in the features of the filamentary structure: The regular filaments become turbulent. This transition goes gradually, as a crossover, with the intermittent behaviour in the region $5 < F < 15$. The character of this change is again common for active nonlinear media [28, 29] as well as for lasers [39, 40].

At Fresnel numbers $F > 15$, the arising filamentary structures become principally different from those existing at low Fresnel numbers. The spatial structures now have no relation to the empty-cavity modes. The modal expansion over the geometrically prescribed Gauss-Laguerre modes is no longer relevant and the boundary conditions have no importance. The laser medium houses a large number of parallel independent filaments exhibiting themselves as a set of bright spots randomly distributed in the transverse cross-section. The number of these random filaments is proportional to F , contrary to the case of low Fresnel numbers, when the number of filaments is proportional to F^2 . The chaotic filaments, being randomly distributed in space, are not correlated with each other. Such a spatio-temporal chaotic behaviour is characteristic of hydrodynamic turbulence, because of which the similar phenomenon in optics is commonly called the *optical turbulence*. In contrast to the regime of low F , where the regularity of spatial structures is prescribed by the cavity geometry and boundary conditions imposing their symmetry constraints, the turbulent optical filamentation is strictly self-organized, with its organization emerging from intrinsic properties of the medium. Since the optical turbulence

is accompanied by the formation of bright filaments with a high density of photons, this phenomenon can be named [35] the *turbulent photon filamentation*. This phenomenon is common for lasers as well as for photorefractive crystals [8, 28, 29].

The first observations of the turbulent filamentary structures in lasers, to my knowledge, were accomplished in the series of experiments [41–45] with the resonatorless superluminescent lasers on the vapours of Ne, Tl, Pb, N₂, and N₂⁺. In these experiments, the typical characteristics were as follows: $\lambda \approx 5 \times 10^{-5}$ cm, $R \approx 0.1 - 0.3$ cm, $L \approx 20 - 50$ cm, and $F \approx 10 - 100$. The number of filaments was $N_f \sim 10^2 - 10^3$, with the typical radius $r_f \approx 0.01$ cm.

The filamentary structures in large-aperture optical devices have been observed in several lasers, as reviewed in [39, 40], and in photorefractive crystals [28, 29]. Numerical simulations have been accomplished [46]. Experimental works mainly dealt with the CO₂ lasers [39, 40, 47, 48], dye lasers [49], and semiconductor lasers [50, 51].

The turbulent nature of filamentation occurring in high Fresnel number lasers was carefully studied in a series of experiments [52–58] with CO₂ lasers and dye lasers. Irregular temporal behaviour was observed in local field measurements. It was found that the transverse correlation length was rather short. Randomly distributed transverse patterns generated in short times were observed, being shot-to-shot nonreproducible. For intermediate Fresnel numbers $F \sim 10$, instantaneous transverse structures were randomly distributed in space, but after being temporally averaged, they displayed a kind of regularity related to the geometrical boundary conditions. This type of combination of irregular instantaneous patterns with the averaged or stationary pattern, showing the remnant ordering, is understandable for the intermediate regime in the crossover region $5 < F < 15$. Fully developed optical turbulence is reached as the Fresnel number increases up to $F \sim 100$.

The typical laser parameters are as follows. The pulsed CO₂ laser, with the wavelength $\lambda = 1.06 \times 10^{-3}$ cm and frequency $\omega = 1.78 \times 10^{14}$ s⁻¹, emits the pulses of $\tau_p \approx 0.7 \times 10^{-7}$ s or 10^{-6} s. The aperture radius $R \approx 1$ cm, laser length $L = 100$ cm. The inversion and polarization decay rates are $\gamma_1 = 10^7$ s⁻¹ and $\gamma_2 = 3 \times 10^9$ s⁻¹. The CO₂ density is $\rho = 2 \times 10^{18}$ cm⁻³. The Fresnel number is $F \approx 10$. The characteristic filament radius is $r_f \approx 0.1$ cm.

The pulsed dye laser, with the wavelength $\lambda = 0.6 \times 10^{-4}$ cm and frequency $\omega = 3.14 \times 10^{15}$ s⁻¹, produces pulses of $\tau_p \approx 0.5 \times 10^{-6}$ s. The decay rates are $\gamma_1 = 4 \times 10^8$ s⁻¹ and $\gamma_2 = 10^{12}$ s⁻¹. The cavity length is $L \approx 20$ cm. By varying the aperture radius between 0.3 cm and 0.8 cm, the Fresnel number can be changed by an order, between $F = 15$ and $F = 110$. The typical filaments radius is $r_f \approx 0.01$ cm.

The theory of turbulent photon filamentation has been developed in [8, 35–38, 59].

Filaments are randomly distributed in the transverse cross-section of the laser cavity, evolving in space and time independently of each other. The characteristics of each filament essentially depend on the value of the related coupling parameter g . For cylindric symmetry the latter can be presented in the form

$$g = 2\pi\rho \frac{\gamma_0}{\gamma_2} \int_0^{R_f} r_{\perp} dr_{\perp} \int_{-L/2}^{L/2} \frac{\sin(k_0\sqrt{r_{\perp}^2 + z^2} - kz)}{k_0\sqrt{r_{\perp}^2 + z^2}} dz . \quad (166)$$

Keeping in mind the resonance condition $k_0 \approx k$ and introducing the variable $x = k(\sqrt{r_{\perp}^2 + z^2} - z)$, we have

$$g = 2\pi \frac{\rho\gamma_0}{k\gamma_2} \int_0^{R_f} r_{\perp} dr_{\perp} \int_{kr_{\perp}^2/L}^{kL} \frac{\sin x}{x} dx . \quad (167)$$

Since $\lambda \ll L$, the upper limit kL in the integral (167) can be replaced by $kL \rightarrow \infty$. This gives

$$g = 2\pi \frac{\rho\gamma_0}{k\gamma_2} \int_0^{R_f} \left[\frac{\pi}{2} - \text{Si} \left(\frac{kr_{\perp}^2}{L} \right) \right] r_{\perp} dr_{\perp} , \quad (168)$$

with the integral sine

$$\text{Si}(x) \equiv \int_0^x \frac{\sin u}{u} du = \frac{\pi}{2} + \int_{\infty}^x \frac{\sin u}{u} du .$$

Introducing the notation

$$\varphi \equiv \frac{\pi R_f^2}{\lambda L} , \quad (169)$$

varying in the interval $0 \leq \varphi \leq \pi F$ and playing the role of an effective Fresnel number for a given filament, we transform equation (168) to

$$g(\varphi) = \pi \frac{\rho\gamma_0 L}{k^2\gamma_2} \left[\pi\varphi - \int_0^{2\varphi} \text{Si}(x) dx \right] . \quad (170)$$

In the same manner, the coupling parameter (145) can be reduced to

$$g'(\varphi) = -\pi \frac{\rho\gamma_0 L}{k^2\gamma_2} \int_0^{2\varphi} \text{Ci}(x) dx , \quad (171)$$

with the integral cosine

$$\text{Ci}(x) \equiv \int_{\infty}^x \frac{\cos u}{u} du .$$

Performing the integration we find

$$g(\varphi) = \pi \frac{\rho\gamma_0 L}{k^2\gamma_2} [\pi\varphi - 2\varphi\text{Si}(2\varphi) + 1 - \cos(2\varphi)] ,$$

$$g'(\varphi) = \pi \frac{\rho\gamma_0 L}{k^2\gamma_2} [\sin(2\varphi) - 2\varphi\text{Ci}(2\varphi)] .$$

Thus, the coupling parameters are functions of the effective variable (169), which, in turn, depends on the enveloping radius R_f related to the effective filament radius r_f . If we assume that the radiation intensity in the transverse cross-section of a filament is distributed by the Gaussian law and if we define the effective filament radius as the mean-square deviation from the filament axis, then we get the relation $r_f = 0.55R_f$.

In general, the filaments of different radii can arise. However, some of them are more stable than others, because of which the overwhelming majority of the filaments will possess the radii close to a typical value. The distribution of filaments with respect to their radii and, hence, the typical radius, can be found by invoking the general method of probabilistic pattern selection [8, 36, 60, 61]. Following this approach, we define the probability distribution

$$p(\varphi, t) = \frac{1}{Z(t)} \exp\{-X(\varphi, t)\} \quad (172)$$

for a filament characterized by the variable φ at the moment of time t . Here

$$X(\varphi, t) = \text{Re} \int_0^t \text{Tr} \hat{J}(\varphi, t') dt' \quad (173)$$

is the *expansion exponent*, expressed through the Jacobian matrix \hat{J} of the evolution equations, and

$$Z(t) = \int \exp\{-X(\varphi, t)\} d\varphi$$

is the normalizing factor. The expansion exponent (173) defines the *local expansion rate*

$$\Lambda(\varphi, t) \equiv \frac{1}{t} X(\varphi, t) . \quad (174)$$

The latter can be represented as the sum of the local Lyapunov exponents. The partial sum of only positive Lyapunov exponents defines the entropy production rate [62], which does not coincide with the local expansion rate (174).

Thus, the probability for the appearance of filaments, characterized by the parameter φ , is given by the probability distribution (172). As is evident, the most probable is the filament with a typical φ satisfying the *principle of minimal expansion* [8, 36, 60, 61]

$$\max_{\varphi} p(\varphi, t) \iff \min_{\varphi} X(\varphi, t) \iff \min_{\varphi} \Lambda(\varphi, t) . \quad (175)$$

This general principle follows from the minimization of the pattern information and can be employed for arbitrary dynamical systems.

The dynamics of turbulent photon filamentation is described by the general evolution equations (164). Calculating the corresponding Jacobian matrix gives

$$\text{Tr} \hat{J}(\varphi, t) = -\gamma_1 - \gamma_3 - 2\gamma_2(1 - gs) ,$$

with $g = g(\varphi)$ and $s = s(t)$. For $t \gg T_1$, the expansion rate can be presented as

$$\Lambda(\varphi, t) \simeq -\gamma_1 - \gamma_3 - 2\gamma_2(1 - gs^*) . \quad (176)$$

Defining the stationary state s^* , we get

$$\begin{aligned} \Lambda(\varphi, t) &\simeq -\gamma_1 - \gamma_3 - 2\gamma_2(1 + |g\zeta|) & (g\zeta \ll -1) , \\ \Lambda(\varphi, t) &\simeq -\gamma_1 - \gamma_3 - 2\gamma_2 \left(1 - \frac{\gamma_1 g \zeta}{\gamma_1 + \gamma_3} \right) & (|g\zeta| \ll 1) , \\ \Lambda(\varphi, t) &\simeq -\gamma_1 - \gamma_3 - \frac{2\gamma_2 \gamma_3}{\gamma_1 g \zeta} & (g\zeta \gg 1) . \end{aligned} \quad (177)$$

The stationary pumping parameter ζ is in the interval $-1 \leq \zeta \leq 1$, depending on the level of pumping. When there is no stationary pumping, $\zeta = -1$. One says that the pumping is weak, if $-1 < \zeta < 0$, and it is strong, if $0 < \zeta < 1$. Keeping in mind that the coupling parameter g is positive, we see that there exist two different cases, when the stationary pumping is weak or absent, $\zeta < 0$, and when it is strong, $\zeta > 0$. According to the principle of minimal expansion (175), the minimum of the expansion rate corresponds to the maximum of $g(\varphi)$ if $\zeta < 0$, and to the minimum of $g(\varphi)$, if $\zeta > 0$. The extrema of $g(\varphi)$ are given by the equation

$$\text{Si}(2\varphi) = \frac{\pi}{2} . \quad (178)$$

In the standard situation of absent or weak pumping, $\zeta < 0$, we have to look for the absolute maximum of $g(\varphi)$. Then equation (178) gives $\varphi = 0.96$. Therefore $R_f = 0.55\sqrt{\lambda L}$, and the *typical filament radius* is

$$r_f = 0.3\sqrt{\lambda L} . \quad (179)$$

The number of filaments can be estimated as $N_f \approx R^2/R_n^2$, which yields

$$N_f \approx 3.3F . \quad (180)$$

The linear dependence of the filament number on the Fresnel number is characteristic of the turbulent photon filamentation.

Note that under strong stationary pumping ($\zeta > 0$), when we need to look for the minimum of $g(\varphi)$, we would have $\varphi = 2.45$, hence, we would obtain $R_f = 0.88\sqrt{\lambda L}$ and $r_f = 0.5\sqrt{\lambda L}$.

Formula (179) for the typical filament can be compared with the radii observed in experiments. Thus, in different vapour lasers [41–45], one has $r_f \approx 0.01$ cm. For the CO₂ laser and dye lasers, it was found [52–58] that $r_f \approx 0.1$ cm and $r_f \approx 0.01$ cm, respectively. All these data are in good agreement with formula (179).

14 Collective liberation of light

There exist systems, called photonic band-gap materials possessing a prohibited band gap, where light cannot propagate. The spontaneous radiation of atoms, with a frequency inside the prohibited band gap, is strongly suppressed [63]. This means that the equation for the population difference of a single atom can be effectively represented as

$$\frac{ds}{dt} = -\gamma_1(s - s_0) .$$

A single initially excited atom remains excited, so that $s = s_0$, for all times $t > 0$.

However, if the density of doped atoms is sufficiently high, coherent interactions may develop [8]. Then atoms can start radiating even inside the prohibited band gap. This collective phenomenon for atoms with the atomic frequency inside the prohibited band gap is termed *collective liberation of light* [8, 64–68].

When the stationary population imbalance of a single atom is $\zeta = s_0$, then the evolution equations for the ensemble of atoms, after the interaction stage, are

$$\begin{aligned} \frac{du}{dt} &= -(i\Omega + \Gamma)u + s\xi , \\ \frac{dw}{dt} &= -2\Gamma w + (u^*\xi + \xi^*u)s , \\ \frac{ds}{dt} &= -g\gamma_2 w - \frac{1}{2}(u^*\xi + \xi^*u)s - \gamma_1(s - s_0) , \end{aligned} \quad (181)$$

where the effective collective width and collective frequency are

$$\Gamma = \gamma_2(1 - gs) , \quad \Omega = \omega_0 + g'\gamma_2 s . \quad (182)$$

These equations are to be understood as characterizing the radiation dynamics in a separate coherent filament of volume $V_f = \pi R_f^2 L \approx \lambda L^2$.

Employing again the scale separation approach, we solve the equation for the fast variable u , substitute the solution into the equations for the slow variables and average the latter equations over fast oscillations and random variables. Using notation (137), we obtain the equations for the slow variables

$$\frac{dw}{dt} = -2\Gamma w + 2\gamma_3 s^2 , \quad \frac{ds}{dt} = -g\gamma_2 w - \gamma_3 s - \gamma_1(s - s_0) . \quad (183)$$

The random variable, representing the matter, can be written [68,69] in the form

$$\xi(t) = \frac{1}{\sqrt{N_0}} \sum_k \gamma_k \left(b_k e^{-i\omega_k t} + b_k^\dagger e^{i\omega_k t} \right) , \quad (184)$$

where b_k are bosonic degrees of freedom describing the matter and ω_k is the related spectrum of collective excitations. For the bosonic operators, one has the averages

$$\begin{aligned} \langle\langle b_k^\dagger b_p \rangle\rangle &= \delta_{kp} n_k , & n_k &\equiv \langle\langle b_k^\dagger b_k \rangle\rangle , \\ \langle\langle b_k b_p^\dagger \rangle\rangle &= \delta_{kp} (1 + n_k) , & \langle\langle b_k b_p \rangle\rangle &= 0 , \end{aligned} \quad (185)$$

and the normalization condition

$$\sum_k n_k = N_{mat} . \quad (186)$$

The matter is characterized by a frequency gap $\Delta_p \equiv \omega_2 - \omega_1$, inside which collective excitations are suppressed, so that

$$\gamma_k = \begin{cases} 0, & \omega_k \in (\omega_1, \omega_2) \\ \gamma, & \omega_k \notin (\omega_1, \omega_2) . \end{cases} \quad (187)$$

Then equations (137) and (184) yield the dynamic attenuation

$$\gamma_3 = \frac{\Gamma}{N_{mat}} \sum_k |\gamma_k|^2 \left[\frac{n_k}{(\omega_k - \Omega)^2 + \Gamma^2} + \frac{1 + n_k}{(\omega_k + \Omega)^2 + \Gamma^2} \right] . \quad (188)$$

For a narrow gap, such that

$$\frac{\Delta_p}{\omega_1} \ll 1 \quad (\Delta_p \equiv \omega_2 - \omega_1) , \quad (189)$$

attenuation (188) simplifies to

$$\gamma_3 \simeq \frac{4\gamma^2\gamma_2(1 - gs)}{\Delta_p^2 + 4\gamma_2^2(1 - gs)^2} . \quad (190)$$

When the coupling and initial excitation are sufficiently weak, such that

$$|gs_0| \ll 1 , \quad (191)$$

then the stationary solutions for the slow variables are

$$\begin{aligned} w^* &\simeq \left(\frac{\gamma_1 s_0}{\gamma_1 + \gamma_3} \right)^2 \frac{\gamma_3}{\gamma_2} \left[1 + \frac{\gamma_1(\gamma_1 - \gamma_3)}{(\gamma_1 + \gamma_3)^2} gs_0 \right] , \\ s^* &\simeq \frac{\gamma_1 s_0}{\gamma_1 + \gamma_3} \left[1 - \frac{\gamma_1 \gamma_3}{(\gamma_1 + \gamma_3)^2} gs_0 \right] \end{aligned} \quad (192)$$

and the dynamic attenuation (190) is

$$\gamma_3 \simeq \frac{4\gamma^2\gamma_2}{\Delta_p^2 + 4\gamma_2^2} \left[1 - \left(1 - \frac{8\gamma_2^2}{\Delta_p^2 + 4\gamma_2^2} \right) gs^* \right] . \quad (193)$$

In the usual situation, when $\gamma \sim \gamma_1 < \gamma_2$, $\delta_p \gg \gamma_2$, we have $\gamma_3 \ll \gamma_1$. Therefore, in the case of weak coupling (191), the stationary excitation remains practically the same as at the beginning, $s^* \approx s_0$, which means that there is no radiation, hence one can say that the light is locked.

If the coupling is strong, but the initial excitation is weak, such that

$$|gs_0| \gg 1, \quad gs_0 < 0, \quad (194)$$

then the stationary solutions are

$$w^* \simeq \frac{\gamma_3 s_0^2}{\gamma_2 |gs_0|}, \quad s^* \simeq s_0 \left(1 - \frac{\gamma_3}{\gamma_1 |gs_0|} \right) \quad (195)$$

and the attenuation is

$$\gamma_3 \simeq \frac{4\gamma^2 \gamma_2 |gs_0|}{\Delta_p^2 + 4\gamma_2^2 (gs_0)^2}. \quad (196)$$

Again, there is no radiation and the light remains locked, since $s^* \approx s_0$.

When the initial excitation and atomic coupling are sufficiently strong, so that

$$gs_0 \gg 1, \quad (197)$$

then the stationary solutions are

$$w^* \simeq \frac{\gamma_1 s_0}{\gamma_2 g}, \quad s^* \simeq \frac{1}{g} \left(1 - \frac{\gamma_3}{\gamma_1 gs_0} \right) \quad (198)$$

and the attenuation is

$$\gamma_3 \simeq - \frac{4\gamma^2 \gamma_2 gs^*}{\Delta_p^2 + 4\gamma_2^2 (gs^*)^2} \simeq - \frac{4\gamma^2 \gamma_2}{\Delta_p^2 + 4\gamma_2^2}. \quad (199)$$

In this case, atoms radiate and de-excite to the low population imbalance

$$\lim_{t \rightarrow \infty} s(t) = s^* \simeq \frac{1}{g} \ll s_0. \quad (200)$$

That is, there happens the collective liberation of light.

15 Pseudospin atomic squeezing

The effect of squeezing allows one to reduce the level of noise when measuring a required quantity. Generally, the notion of squeezing is introduced for two operators, say \hat{A} and \hat{B} . The uncertainty in measuring the observable quantity, corresponding to the operator \hat{A} , is characterized by the operator dispersion or variance

$$\text{var}(\hat{A}) \equiv \langle \hat{A}^+ \hat{A} \rangle - |\langle \hat{A} \rangle|^2. \quad (201)$$

The squeezing of \hat{A} with respect to \hat{B} is defined by the *squeezing factor*

$$Q(\hat{A}, \hat{B}) \equiv \frac{2\text{var}(\hat{A})}{|\langle [\hat{A}, \hat{B}] \rangle|}. \quad (202)$$

The Heisenberg uncertainty relation

$$\text{var}(\hat{A})\text{var}(\hat{B}) \geq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2 \quad (203)$$

can be written as

$$Q(\hat{A}, \hat{B})Q(\hat{B}, \hat{A}) \geq 1 . \quad (204)$$

One says that \hat{A} is squeezed with respect to \hat{B} if

$$\text{var}(\hat{A}) < \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle| , \quad (205)$$

which implies that

$$Q(\hat{A}, \hat{B}) < 1 . \quad (206)$$

Dealing with the pseudospin operators

$$S_N^z \equiv \sum_{j=1}^N S_j^z , \quad S_N^- \equiv \sum_{j=1}^N S_j^- , \quad (207)$$

we derive the evolution equations for the variables

$$\begin{aligned} u &\equiv \frac{2}{N} \sum_{j=1}^N \langle S_j^- \rangle = \frac{2}{N} \langle S_N^- \rangle , \\ w &\equiv \frac{4}{N^2} \sum_{i \neq j}^N \langle S_i^+ S_j^- \rangle = |u|^2 , \\ s &\equiv \frac{2}{N} \sum_{j=1}^N \langle S_j^z \rangle = \frac{2}{N} \langle S_N^z \rangle . \end{aligned} \quad (208)$$

Taking into account the properties

$$\text{var}(S_N^z) = \frac{N}{4} (1 - s^2) , \quad \langle [S_N^z, S_N^-] \rangle = \frac{N}{2} |u| , \quad (209)$$

we obtain the squeezing factor

$$Q(S_N^z, S_N^-) = \frac{1 - s^2}{\sqrt{w}} , \quad (210)$$

characterizing the squeezing of S_N^z with respect to S_N^- . If this factor is less than one, this means that measuring the atomic imbalance s can be done with a better accuracy than measuring the transition quantities, such as, e.g., coherence intensity w .

Different regimes of atomic evolution result in a variety of types of temporal behaviour of the squeezing factor (210). Its behaviour also depends on whether the vacuum, describing the matter, is squeezed. The level of squeezing can be regulated in the process of punctuated superradiance [70].

16 Operator entanglement production

The notion of entanglement is nowadays widely studied because of its role in quantum information processing and quantum computing [71–73]. It is necessary to distinguish entanglement from entanglement production [74, 75]. Entanglement characterizes the state of a bipartite, or more generally, of a many-body system [71–73], while entanglement production shows the amount of entanglement produced by an operation associated with an operator [74, 75]. Here we consider entanglement production that plays an important role in both quantum information processing and quantum measurements [76].

Suppose we need to find out how much entanglement is produced by an operator \hat{A} . The operator acts on a multidimensional Hilbert space

$$\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i . \quad (211)$$

Such spaces are typical of many-body systems [77, 78]. The operator, acting on a non-entangled state, such as

$$\varphi = \bigotimes_{i=1}^N \varphi_i \quad (\varphi_i \in \mathcal{H}_i) ,$$

can transfer it to an entangled state.

The action of an operator has to be compared with the action of its non-entangling counterpart that is defined as

$$\hat{A}^\otimes \equiv \frac{\bigotimes_{i=1}^N \hat{A}_i}{(\text{Tr}_{\mathcal{H}} \hat{A})^{N-1}} , \quad (212)$$

where

$$\hat{A}_i \equiv \text{Tr}_{\mathcal{H}/\mathcal{H}_i} \hat{A} \quad (213)$$

is a factor-operator obtained by tracing out from \hat{A} all degrees of freedom except one associated with the subspace \mathcal{H}_i . The denominator in equation (212) is chosen so that to preserve the normalization condition

$$\text{Tr}_{\mathcal{H}} \hat{A}^\otimes = \text{Tr}_{\mathcal{H}} \hat{A} . \quad (214)$$

The measure of entanglement production is defined as

$$\varepsilon(\hat{A}) \equiv \log \frac{\|\hat{A}\|}{\|\hat{A}^\otimes\|} , \quad (215)$$

where $\|\hat{A}\|$ implies an operator norm. It is possible to choose different definitions of the norm. One possibility would be to opt for the Hilbert-Schmidt norm

$$\|\hat{A}\| \equiv \sqrt{\text{Tr}_{\mathcal{H}}(\hat{A}^\dagger \hat{A})} , \quad (216)$$

whose advantage is that it does not depend on the basis used for calculating the trace.

Generally, the operator \hat{A} can depend on some parameters, so that one can study the dependence of measure (215) on this parameter. In particular, this parameter can be time [79], hence, it is possible to consider temporal behaviour of entanglement production, for instance in the process of atomic collective radiation [80].

Another possibility to investigate the evolutional entanglement production is by considering the entanglement production due to the evolution operator

$$\hat{U}(t) = e^{-i\hat{H}t} ,$$

in which \hat{H} is the system Hamiltonian. Following the general way, we define the partial factor operator

$$\hat{U}_i(t) \equiv \text{Tr}_{\mathcal{H}/\mathcal{H}_i} \hat{U}(t) \quad (217)$$

and construct the corresponding non-entangling evolution operator

$$\hat{U}^\otimes(t) = \frac{\bigotimes_{i=1}^N \hat{U}_i(t)}{[\text{Tr}_{\mathcal{H}} \hat{U}(t)]^{N-1}} . \quad (218)$$

The measure of the evolutional entanglement production reads

$$\varepsilon(t) \equiv \varepsilon(\hat{U}(t)) = \log \frac{\|\hat{U}(t)\|}{\|\hat{U}^\otimes(t)\|} . \quad (219)$$

For the Hilbert-Schmidt norms, we have

$$\|\hat{U}(t)\|^2 = \prod_{i=1}^N d_i \quad (d_i \equiv \dim \mathcal{H}_i) ,$$

$$\|\hat{U}^\otimes(t)\|^2 = \frac{\prod_{i=1}^N \|\hat{U}_i(t)\|_{\mathcal{H}_i}^2}{|\text{Tr}_{\mathcal{H}} \hat{U}(t)|^{2(N-1)}} .$$

Therefore the evolutional entanglement production is characterized by the measure

$$\varepsilon(t) = \frac{1}{2} \log \left\{ \left| \text{Tr}_{\mathcal{H}} \hat{U}(t) \right|^{2(N-1)} \prod_{i=1}^N \frac{d_i}{\|\hat{U}_i(t)\|_{\mathcal{H}_i}^2} \right\} . \quad (220)$$

This measure shows to what extent the evolution operator produces entanglement.

17 Conclusion

A general approach is presented that can be used for both atomic as well as spin systems. The approach is based on the pseudospin representation of evolution equations and employs the methods of stochastic quantization, scale separation, transverse mode expansion, and probabilistic pattern selection. The generality of the approach is also in the possibility of treating all stages of evolution, which makes it possible to study the self-organization of coherence from initial chaotic fluctuations. The qualitatively different temporal intervals are: interaction stage, chaotic stage, coherent stage, relaxation stage, and quasi-stationary stage. The regimes of pure superradiance, triggered superradiance, pulsing superradiance, and punctuated superradiance are analyzed.

This approach has earlier been used for describing nonequilibrium coherent phenomena in spin systems [6–9, 14, 74, 75]. In the present paper, it is demonstrated that the same mathematical techniques are applicable to characterizing radiation processes in atomic systems. The

approach is illustrated by describing several interesting effects, such as triggering dipolar waves, turbulent photon filamentation, collective liberation of light, pseudospin atomic squeezing, and operator entanglement production.

Although physical processes in spin and atomic systems are rather different [6, 13–16], the mathematical methods of investigation turns out to be the same, which is the advantage of the suggested approach.

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